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A new BEM–FEM coupling strategy for two-dimensional fluid–solid interaction problems

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Abstract

We present a numerical method to solve a fluid–solid interaction problem posed in the plane. In this scheme, we use a finite element method to approximate the solid vibrations and the near wave field. The far field effects are taken into account by means of boundary integral equations posed on an artificial interface that contains the obstacle. The boundary unknown involved in our formulation is approximated by a spectral method. We obtain a fully discrete Galerkin procedure whose main advantage is the simplicity of the quadratures used to approximate the weakly singular boundary integrals. We provide numerical results that illustrate the accuracy of our method and the stability of the algorithm used to solve the linear systems of equations that arise from this discretization technique.

Keywords: Exterior boundary value problem; Helmholtz equation; Elastodynamic equation; Integral equations; Finite elements; Spectral methods

1. Introduction

We consider a bounded elastic body (the obstacle) embedded in an unbounded compressible inviscid fluid (the acoustic medium). Any acoustic wave incident on the obstacle transmits part of its energy in the form of elastic vibrations. At the same time, the elastic vibrations of the solid cause acoustic waves in the fluid. In this paper, we introduce a numerical scheme to compute the scattered waves and the elastic vibrations that take place in this interaction between the fluid and the solid.

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The numerical difficulties related to the fact that the scattered wave propagates in an unbounded region is overcomed by imposing *absorbing boundary conditions* on an artificial boundary containing the obstacle. This permits one to incorporate the far-field effects into a finite element discretization of the problem in a bounded region. The absorbing boundary conditions may be of local (differential) or global type; we refer to [3,5] for a review of such methods.

In this paper, we use linear integral equations as nonlocal boundary conditions on the artificial interface. This strategy gives rise to numerical schemes based on a combination of a finite element method (FEM) and a boundary element method (BEM). Here, we follow [4] and propose the so-called symmetric BEM–FEM formulation due to Costabel [2] to solve the fluid–solid interaction problem. We point out that Bielak and MacCamy propose in [1] a BEM–FEM method that also leads to a symmetric formulation and that avoids the use of the hypersingular integral operator, which is the integral operator whose kernel has the most severe singularity. In [1,4], the interface that separates the two mediums (the wet interface) is used as a coupling boundary. In this case, the well-posedness of the resulting formulation (at the continuous level) requires regularity assumptions that may not be fulfilled in practice by the wet interface. Here, we impose the absorbing boundary conditions on a smooth but arbitrary interface that contains the obstacle in its interior. This enlarges a little the domain of finite element computations but this drawback is compensated by the fact that we remove the limitation to problems with smooth wet boundaries.

The presence of integrals with nearly singular integrands augurs that the matrix assembly process is a delicate operation in all the BEM–FEM coupling procedures. The design of efficient algorithms for this task is of great importance in order to improve the practicability of these methods. Another handicap related to this kind of approximation methods concerns the complicated linear systems of equations to which they lead. The corresponding matrices are general (symmetric in the case of symmetric BEM–FEM formulations) and their sparsity is reduced by the coupling procedure. Neither of these two difficulties are addressed in [1,4]. Here, we will show how to take advantage of the techniques developed in [9–14] in order to handle these drawbacks in the case of a two-dimensional BEM–FEM formulation of a solid–fluid interaction problem.

Recently, the classical BEM–FEM formulations have been rewritten (see [9–14]) by changing all terms on the interface to periodic functions by means of a smooth parameterization of the artificial boundary. These new formulations allow one to approximate the weakly singular boundary integrals by elementary quadrature formulas. Furthermore, as shown in [13], they permit one to approximate the periodic representation of the unknown defined on the boundary by trigonometric polynomials.

The advantage of such a hybrid scheme that combines a finite element method with a spectral method is that few degrees of freedom are needed on the interface boundary as we confirm by our numerical experiments. This permits one to eliminate the periodic unknown at matricial level by a static condensation process and reduce in the way the complexity of the linear systems. Here we use a preconditioned GMRES method to solve the reduced linear system of equations whose matrices are complex symmetric but not definite. The resulting iterative method only requires the solution of standard (interior) elliptic finite element problems. It also allows one to avoid storing the huge global matrix. Our numerical experiments reveal that the number of iterations of the algorithm does not increase with the number of unknowns.

The paper is organized as follows. In Section 2 we give a more detailed description of the physical assumptions and we set up the governing equations. We derive in Section 3 a variational formulation of the problem by using integral equations as nonlocal boundary conditions on a smooth artificial interface. We also state a theorem on the uniqueness of solution of the resulting problem. In Sections 4 and 5, we introduce a discrete problem and provide numerical quadratures that permits one to write a full discretization of the equations. Finally, in Section 6, we present our numerical results together with the iterative method used to solve the systems of linear equations.

In the sequel, we deal with complex valued functions and the symbol $\imath$ is used for $\sqrt{-1}$. We denote by $\overline{\alpha}$ the conjugate of a complex number $\alpha \in \mathbb{C}$ and by $|\alpha|$ its modulus. Small boldface letters will denote vectors or vector valued functions.

2. Physical assumptions and governing equations

We are concerned with the interaction between an elastic body and a fluid that fills the space around it. We suppose that a wave is incident upon the body and we are required to determine its response and the scattered wave.

We assume that the obstacle is an infinitely long cylinder parallel to the x_3-axis whose cross-section is Ω_s. We denote by Σ the boundary of Ω_s. The incident acoustic wave and the volume force acting on the obstacle are suppose to exhibit a time-harmonic behavior with frequency ω. We will denote their amplitudes $w = w(x_1, x_2)$ and $\mathbf{f} = \mathbf{f}(x_1, x_2)$, respectively. The incident wave is generally taken to satisfy the Helmholtz equation $\Delta w + k^2 w = 0$ in $\Omega_f := \mathbb{R}^2 \setminus \overline{\Omega_s}$.

The phenomenon is invariant under a translation in the x_3-direction. Then, we may consider a bidimensional model posed in the frequency domain. The unknowns of the problem are the amplitude $\mathbf{u} : \Omega_s \to \mathbb{C}^2$ of the solid displacements field and the amplitude $p : \Omega_f \to \mathbb{C}$ of the scattered pressure.

We suppose that the solid is isotropic and linearly elastic, with mass density ρ_s and Lamé moduli λ, μ. We denote as usual the stress tensor by $\sigma(\mathbf{u}) := \lambda \operatorname{tr} \varepsilon(\mathbf{u}) I + 2\mu\varepsilon(\mathbf{u})$, where $\varepsilon_{ij}(\mathbf{u}) := \frac{1}{2}(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})$ is the infinitesimal strain tensor. Furthermore, we assume that the fluid is ideal, compressible and homogeneous with mass density ρ_f and wave number $k = \frac{\omega}{c}$ where c is the speed of sound in the linearized fluid.

Let us denote by $\mathbf{n}$ the unit normal on Σ directed into Ω_f. Under the hypothesis of small oscillations both in the solid and the fluid, $\mathbf{u}$ and p are found out to satisfy the equations

$$\begin{aligned}
&\nabla \cdot \sigma(\mathbf{u}) + \rho_s \omega^2 \mathbf{u} = -\mathbf{f} \quad \text{in } \Omega_s,\\
&\Delta p + k^2 p = 0 \quad \text{in } \Omega_f,\\
&\sigma(\mathbf{u})\mathbf{n} = -(p + w)\mathbf{n} \quad \text{on } \Sigma,\\
&\rho_f \omega^2 \mathbf{u} \cdot \mathbf{n} = \frac{\partial (p + w)}{\partial \mathbf{n}} \quad \text{on } \Sigma
\end{aligned} \tag{1}$$

and the decay conditions

$$p = \mathrm{O}(r^{-1/2}), \quad \frac{\partial p}{\partial r} - \imath k p = \mathrm{o}(r^{-1/2}) \tag{2}$$

when $r \to +\infty$ uniformly for all directions $\frac{x}{|x|}$.

The first two equations of (1) are, respectively, the *elastodynamic* and *acoustic* equations in time-harmonic regime. The transmission conditions posed on Σ represent the equilibrium of forces (*dynamic boundary condition*) and the equality of the normal displacements of solid and fluid (*kinematic boundary condition*). Finally, Eq. (2) means that the far field absorbs the outgoing waves (cf. [5] for more details).

It is known that if $\mathbf{f} = \mathbf{0}$ and $w = 0$ then $p = 0$ and $\mathbf{u}$ is solution of (see [7])

$$\begin{aligned}
&\nabla \cdot \sigma(\mathbf{u}) + \rho_s \omega^2 \mathbf{u} = 0 \quad \text{in } \Omega_s,\\
&\sigma(\mathbf{u})\mathbf{n} = 0 \quad \text{on } \Sigma,\\
&\mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } \Sigma.
\end{aligned} \tag{3}$$

It turns out that for certain regions and some frequencies $\rho_s \omega^2$, known as *Jones frequencies*, problem (3) have nontrivial solutions. This seems to be a rare eventuality but we will, in any case, assume that (3) admits only the trivial solution.

3. A variational formulation with nonlocal boundary conditions

Let us introduce an artificial boundary Γ such that Ω_s lays in its interior. Then, Γ separates $\mathbb{R}^2$ into a bounded domain Ω^- and an unbounded region Ω_f^+ exterior to Γ. We denote $\Omega_f^- := \Omega_f \cap \Omega^-$. Notice that $\overline{\Omega^-} = \overline{\Omega_s} \cup \overline{\Omega_f^-}$; cf. Fig. 1.

We consider the sesquilinear forms

$$A(\mathbf{u},\mathbf{v}) := \int_{\Omega_s} (\sigma(\mathbf{u}) : \varepsilon(\mathbf{v}) - \rho_s \omega^2 \mathbf{u}\cdot\mathbf{v})\,\mathrm{d}\boldsymbol{x},$$

$$a(p,q) := \frac{1}{\rho_f \omega^2} \int_{\Omega_f^-} (\nabla p \cdot \nabla q - k^2 pq)\,\mathrm{d}\boldsymbol{x} \quad \text{and} \quad D(\mathbf{v},q) := \int_\Sigma \mathbf{v}\cdot\mathbf{n} q\,\mathrm{d}\tau.$$

It is straightforward to show that in Ω_s $\mathbf{u}$ satisfies the variational formulation

$$\text{find } \mathbf{u} \in (H^1(\Omega_s))^2 \text{ such that } A(\mathbf{u},\mathbf{v}) + D(\mathbf{v},p) = L(\mathbf{v}) \quad \forall \mathbf{v} \in (H^1(\Omega_s))^2, \tag{4}$$

where

$$L(\mathbf{v}) := \int_{\Omega_s} \mathbf{f}\cdot\mathbf{v}\,\mathrm{d}\boldsymbol{x} - D(\mathbf{v},w)$$

while $p|_{\Omega_f^-}$ is a solution of

$$\text{find } p \in H^1(\Omega_f^-) \text{ such that } a(p,q) + D(\mathbf{u},q) - \frac{1}{\rho_f\omega^2}\int_\Gamma \frac{\partial p}{\partial \boldsymbol{v}} q\,\mathrm{d}\tau = \ell(q) \quad \forall q \in H^1(\Omega_f^-). \tag{5}$$

Here, the unit normal $\boldsymbol{v}$ on Γ is directed into Ω_f^+ and

$$\ell(q) := \frac{1}{\rho_f\omega^2}\int_\Sigma \frac{\partial w}{\partial \mathbf{n}} q\,\mathrm{d}\tau.$$

On the other hand, using a Green formula, the radiation conditions (2) and the fact that p solves the Helmhotz equation in Ω_f^+, one arrives at the following integral representation:

$$p(\boldsymbol{x}) = \int_\Gamma \frac{\partial E(\boldsymbol{x},\boldsymbol{y})}{\partial \boldsymbol{v}_y} p(\boldsymbol{y})\,\mathrm{d}\tau_y - \int_\Gamma E(\boldsymbol{x},\boldsymbol{y}) \frac{\partial p}{\partial \boldsymbol{v}}(\boldsymbol{y})\,\mathrm{d}\tau_y \quad \forall \boldsymbol{x} \in \Omega_f^+, \tag{6}$$

where

$$E(\boldsymbol{x},\boldsymbol{y}) := \frac{\imath}{4} H_0^{(1)}(k|\boldsymbol{x}-\boldsymbol{y}|)$$

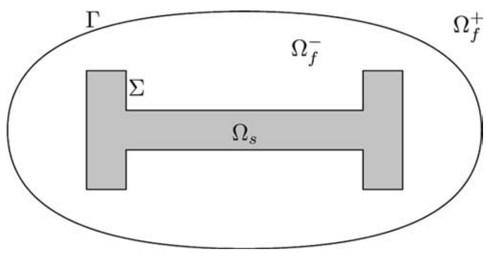

Fig. 1. Geometry of the problem.

is the radial outgoing fundamental solution of the Helmholtz equation and $H_0^{(1)}$ stands for the Hankel function of order 0 and first kind. The symmetric BEM–FEM method introduced in [2] uses two boundary integral identities relating on Γ the trace of p and its normal derivative $\frac{\partial p}{\partial \nu}$. These boundary integral equations arise from the integral representation formula (6) and the jump conditions of the layer potentials. Our purpose is to perform the coupling of these boundary equations with (4) and (5), but let us first introduce some notations and basic properties.

In the sequel, we choose Γ to be an infinitely differentiable boundary and we denote by $\mathbf{x} : \mathbb{R} \to \mathbb{R}^2$ a regular 2π-periodic parametric representation of this curve

$$|\mathbf{x}'(s)| > 0 \quad \forall s \in \mathbb{R} \quad \text{and} \quad \mathbf{x}(s) = \mathbf{x}(t) \Longleftrightarrow t - s \in 2\pi\mathbb{Z}.$$

Therefore, we can identify any function q defined on Γ with the 2π-periodic function $q \circ \mathbf{x}$. This parameterization of Γ also allows us to define the parameterized trace on Γ as the unique extension of

$$\gamma : \mathscr{C}^\infty(\overline{\Omega_f^-}) \to L^2(0, 2\pi),$$

$$q \mapsto \gamma q := q|_\Gamma \circ \mathbf{x}$$

to the whole of $H^1(\Omega_f^-)$. Theorem 8.15 of [6] proves that $\gamma : H^1(\Omega_f^-) \to H^{1/2}$ is bounded and onto, where $H^{1/2}$ is the completion of $\mathscr{C}_{2\pi}^\infty$ with the norm

$$\|g\|_{1/2} := \left(\sum_{n \in \mathbb{Z}} (1 + n^2)^{1/2} |\widehat{g}(n)|^2 \right)^{1/2}.$$

We denoted here by $\mathscr{C}_{2\pi}^\infty$ the space of 2π-periodic and infinitely differentiable complex valued functions of a single variable and

$$\hat{g}(n) := \frac{1}{2\pi} \int_0^{2\pi} g(s) e^{-\imath ns} \, \mathrm{d}s$$

are the Fourier coefficients of $g \in \mathscr{C}_{2\pi}^\infty$. We will denote by $H^{-1/2}$ the dual space of $H^{1/2}$. The $L^2(0, 2\pi)$-bilinear form product $\int_0^{2\pi} \lambda(s)\mu(s) \, \mathrm{d}s$ can be extended to represent the duality between $H^{-1/2}$ and $H^{1/2}$. We will keep the same notation for this duality bracket.

We introduce parameterized versions of the single and double layer acoustic potentials

$$\mathscr{S}g(s) := \int_0^{2\pi} V(s, t) g(t) \, \mathrm{d}t \quad \text{and} \quad \mathscr{D}g(s) := \int_0^{2\pi} K(s, t) g(t) \, \mathrm{d}t,$$

where

$$V(s, t) := \frac{\imath}{4} H_0^{(1)}(k|\mathbf{x}(s) - \mathbf{x}(t)|)$$

and

$$K(s, t) := -\frac{k\imath}{4} H_1^{(1)}(k|\mathbf{x}(t) - \mathbf{x}(s)|) \frac{x_2'(t)(x_1(t) - x_1(s)) - x_1'(t)(x_2(t) - x_2(s))}{|\mathbf{x}(t) - \mathbf{x}(s)|},$$

with $H_1^{(1)}$ being the Hankel function of first kind and order one.

Let us introduce the auxiliary unknown ξ given in terms of the normal derivative of p on Γ by

$$\xi := |\mathbf{x}'|\frac{\partial p}{\partial \boldsymbol{v}}\circ\mathbf{x}.$$

Parameterizing the integrals on Γ in the traditional symmetric BEM–FEM method (cf. [2]) yields to (a similar strategy is used in [9,11,15])

$$\begin{aligned} \gamma p &= \left(\frac{1}{2}\mathscr{I} + \mathscr{D}\right)\gamma p - \mathscr{S}\xi, \\ \xi &= -\mathscr{H}\gamma p + \left(\frac{1}{2}\mathscr{I} - \mathscr{D}^*\right)\xi, \end{aligned} \tag{7}$$

where $\mathscr{I}$ is the identity operator, $\mathscr{D}^*$ is the adjoint of $\mathscr{D}$ and $\mathscr{H}$ is the hypersingular operator which is related to the single layer operator via tangential derivatives, see [8]. With our notations this relation reads

$$\int_0^{2\pi} \eta(\mathscr{H}\psi)\,\mathrm{d}t = \int_0^{2\pi} \eta'(\mathscr{S}\psi')\,\mathrm{d}t - k^2\int_0^{2\pi}\eta(\tilde{\mathscr{S}}\psi)dt \quad \forall\psi,\eta\in H^{1/2},$$

where $\tilde{\mathscr{S}}$ is the integral operator whose kernel is given by $\tilde{V}(t,s) := \mathbf{x}'(t)\cdot\mathbf{x}'(s)V(t,s)$.

Combining (4) and (5) with (7) we arrive at the following global weak formulation of (1) and (2):

$$\begin{aligned} &\text{find } \mathbf{u}\in(H^1(\Omega_s))^2,\ p\in H^1(\Omega_f^-) \text{ and } \xi\in H^{-\frac{1}{2}} \text{ such that} \\ &\quad A(\mathbf{u},\mathbf{v}) + D(\mathbf{v},p) = L(\mathbf{v}), \\ &\quad a(p,q) + D(\mathbf{u},q) - b(\gamma q,\xi) + c((\gamma p)',(\gamma q)') - k^2 d(\gamma p,\gamma q) = \ell(q), \\ &\quad -b(\gamma p,\eta) - c(\xi,\eta) = 0 \end{aligned} \tag{8}$$

for all $\mathbf{v}\in(H^1(\Omega_s))^2$, $q\in H^1(\Omega_f^-)$ and $\eta\in H^{-\frac{1}{2}}$. We have denoted

$$c(\xi,\eta) := \frac{1}{\rho_f\omega^2}\int_0^{2\pi}\eta(t)(\mathscr{S}\xi)(t)\,\mathrm{d}t \quad\text{and}\quad b(\gamma p,\eta) := \frac{1}{\rho_f\omega^2}\int_0^{2\pi}\eta(t)\left(\frac{1}{2}\mathscr{I}-\mathscr{D}\right)(\gamma p)(t)\,\mathrm{d}t$$

and

$$d(\gamma p,\gamma q) := \frac{1}{\rho_f\omega^2}\int_0^{2\pi}\gamma q(t)(\tilde{\mathscr{S}}\gamma p)(t)\,\mathrm{d}t.$$

Theorem 1. *Assume that problem* (3) *admits only the trivial solution and that* k^2 *is not an eigenvalue of the Laplacian in* Ω *with a Dirichlet boundary condition on* Γ. *Then, the solution of problem* (8) *is unique.*

Proof 1. Let $(\mathbf{u}_0,p_0,\xi_0)$ be a solution of (8) with $\mathbf{f}=\mathbf{0}$ and $w=0$. We define the function

$$\tilde{p}(\boldsymbol{x}) := \begin{cases} p_0(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega_f^-, \\ z(\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\Omega_f^+, \end{cases}$$

where

$$z(\boldsymbol{x}) := \int_0^{2\pi}\frac{\partial E}{\partial \boldsymbol{v}_y}(\boldsymbol{x},\mathbf{x}(t))\gamma p_0(t)|\mathbf{x}'(t)|\,\mathrm{d}t - \int_0^{2\pi}E(\boldsymbol{x},\mathbf{x}(t))\xi_0(t)\,\mathrm{d}t.$$

It is easy to show that $\mathbf{u}_0$, p_0 and ξ_0 solve the equations:

$$\begin{aligned}
&\nabla \cdot \sigma(\mathbf{u}_0) + \rho_s \omega^2 \mathbf{u}_0 = \mathbf{0} \quad \text{in } \Omega_s,\\
&\Delta p_0 + k^2 p_0 = 0 \quad \text{in } \Omega_f^-,\\
&\sigma(\mathbf{u}_0)\mathbf{n} = -(p_0 + w)\mathbf{n} \quad \text{on } \Sigma,\\
&\rho_f \omega^2 \mathbf{u}_0 \cdot \mathbf{n} = \frac{\partial(p_0 + w)}{\partial \mathbf{n}} \quad \text{on } \Sigma,\\
&\gamma p_0 = \left(\frac{1}{2}\mathscr{I} + \mathscr{D}\right)\gamma p_0 - \mathscr{S}\xi_0,\\
&|\mathbf{x}'(t)| \frac{\partial p_0}{\partial \nu} \circ \mathbf{x} = -\mathscr{H}\gamma p_0 + \left(\frac{1}{2}\mathscr{I} - \mathscr{D}^*\right)\xi_0.
\end{aligned} \tag{9}$$

On the other hand, z also solves the Helmholtz equation in Ω_f^+

$$\Delta z + k^2 z = 0 \quad \text{in } \Omega_f^+ \tag{10}$$

and it satisfies the asymptotic conditions (2). Besides, the jump properties of the double layer potential and the normal derivative of the single layer potential through Γ provides the relations (cf. [15]):

$$\begin{aligned}
&\gamma z = \left(\frac{1}{2}\mathscr{I} + \mathscr{D}\right)\gamma p_0 - \mathscr{S}\xi_0,\\
&|\mathbf{x}'(t)| \frac{\partial z}{\partial \boldsymbol{\nu}} \circ \mathbf{x} = -\mathscr{H}\gamma p_0 + \left(\frac{1}{2}\mathscr{I} - \mathscr{D}^*\right)\xi_0.
\end{aligned} \tag{11}$$

Combining (10), (11), and (2) with (9) proves that $(\mathbf{u}_0, \tilde{p})$ is a solution of (1) with data $\mathbf{f} = \mathbf{0}$ and $w = 0$. Now, our assumption on problem (3) ensures that $(\mathbf{u}_0, \tilde{p})$ vanishes identically and consequently

$$\left(\frac{1}{2}\mathscr{I} - \mathscr{D}^*\right)\xi_0 = 0.$$

Finally Theorem 3.3.4. of [15] proves that, under our hypothesis on k, operator $\frac{1}{2}\mathscr{I} - \mathscr{D}^*$ is one-to-one and the result follows. $\square$

Remark 1. Standard arguments permits one to show that problem (8) is a compact perturbation of a well-posed problem. Thus, by virtue of the Fredholm alternative, Theorem 1 is in fact also an existence result.

4. Discrete problem

For simplicity of exposition, in the rest of the paper we assume that Σ is a polygonal boundary. Let N be a given integer. We consider the equidistant subdivision $\{t_i := i\pi/N;\ i = 0, \ldots, 2N-1\}$ of the interval $[0, 2\pi]$ with $2N$ grid points. We denote by Ω_h the polygonal domain whose vertices lying on Γ are $\{\mathbf{x}(t_i) : i = 0, \ldots, 2N-1\}$. Let $\{\tau_h\}$ be a regular family of triangulations of $\overline{\Omega_h}$ by triangles T of diameter h_T not greater than $\max|\mathbf{x}'(s)|h$ with $h := \pi/N$. We assume that the restriction $\tau_h^s := \{T \in \tau_h; T \subset \overline{\Omega_s}\}$ of τ_h to $\overline{\Omega_s}$ is a triangulation and set $\tau_h^f := \tau_h \setminus \tau_h^s$. Notice that $\Omega_{f,h}^- := \text{interior}(\cup_{T \in \tau_h^f} T)$ is a polygonal approximation of Ω_f^-.

We introduce the finite element spaces

$$V_h^s := \{v \in \mathscr{C}^0(\overline{\Omega_s});\ v|_T \in P_1(T)\ \forall T \in \tau_h^s\}$$

and

$$V_h^f := \{q \in \mathscr{C}^0(\overline{\Omega_{f,h}^-});\ q|_T \in P_1(T)\ \forall T \in \tau_h^f\},$$

where $P_1(T)$ is the space of linear functions on T.

Let Γ_h be the exterior boundary of $\Omega_{f,h}^-$. We follow [12] and define a discrete counterpart γ_h of the parameterized trace operator γ. This discrete linear operator will relate the space of traces $V_h^f(\Gamma_h) := \{q|_{\Gamma_h}; q \in V_h^f\}$ of functions in V_h^f to the subspace $T_h \subset H^{1/2}$ defined by the set of continuous, 2π-periodic and piecewise linear functions on the uniform partition of $[0, 2\pi]$ into $2N$ grid points. It is clear that

$$\gamma_h : V_h^f(\Gamma_h) \to T_h$$

$$q|_{\Gamma_h} \mapsto \gamma_h q$$

is uniquely determined by the conditions $\gamma_h q(t_i) := q(\mathbf{x}(t_i))$ for $i = 0, \ldots, 2N-1$.

Let n be a given integer and consider the $2n$-dimensional space

$$T_n := \left\{ \sum_{j=0}^{n} a_j \cos jt + \sum_{j=1}^{n-1} b_j \sin jt;\ a_j, b_j \in \mathbb{C} \right\}.$$

The discrete version of (8) is then given by

$$\begin{aligned}
&\text{find } \mathbf{u}_h \in (V_h^s)^2,\ p_h \in V_h^f \text{ and } \xi_n \in T_n \text{ such that} \\
&\quad A(\mathbf{u}_h, \mathbf{v}) + D(\mathbf{v}, p_h) = L(\mathbf{v}), \\
&\quad a(p_h, q) + D(\mathbf{u}_h, q) - b(\gamma_h q, \xi_n) + c((\gamma_h p_h)', (\gamma_h q)') - k^2 d(\gamma_h p_h, \gamma_h q) = \ell(q), \\
&\quad - b(\gamma_h p_h, \eta) - c(\xi_n, \eta) = 0
\end{aligned} \tag{12}$$

for all $\mathbf{v} \in (V_h^s)^2$, $q \in V_h^f$ and $\eta \in T_n$.

5. Full discretization of the equations

5.1. Approximation of the interior terms

Under the condition that ρ_s and ρ_f are constant, the integrals involved in the sesquilinear forms $A(\mathbf{u}, \mathbf{v})$, $a(p, q)$ and $D(\mathbf{v}, q)$ may be computed exactly for discrete variables $\mathbf{u}$, $\mathbf{v}$, p and q.

We can associate to any continuous function $g : \Sigma \to \mathbb{C}$ the continuous and piecewise linear function $I_\Sigma^h(g) : \Sigma \to \mathbb{C}$ uniquely determined by the conditions: $I_\Sigma^h g(\mathbf{a}) = g(\mathbf{a})$ for all vertex $\mathbf{a} \in \tau_h$ that belongs to Σ. We assume that $\mathbf{f}$ is continuous in $\overline{\Omega_s}$ and approximate $L(\mathbf{v})$ for all $\mathbf{v} \in V_h^s$ by

$$L_h(\mathbf{v}) := \sum_{T \in \tau_h^s} \frac{\text{measure}(T)}{3} \sum_{i=1}^{3} (\mathbf{f} \cdot \mathbf{v})(\mathbf{a}_i^T) - \int_\Sigma \mathbf{v} \cdot \mathbf{n} I_\Sigma^h(w)\, \mathrm{d}\tau,$$

where $\mathbf{a}_i^T$ are the vertices of triangle T. We also set

$$\ell(q) \simeq \ell_h(q) := \frac{1}{\rho_f \omega^2} \int_\Sigma I_\Sigma^h\left(\frac{\partial w}{\partial \mathbf{n}}\right) q\, \mathrm{d}\tau$$

for all $q \in V_h^f$.

5.2. Approximation of $c(\cdot,\cdot)$

For any continuous and 2π-periodic function g we consider the composite trapezoidal rule

$$\mathcal{Q}_n(g) := \frac{\pi}{n}\sum_{i=0}^{2n-1} g\left(\frac{i\pi}{n}\right)$$

associated to the uniform partition of $[0,2\pi]$ into $2n$ grid points.

We proceed as in [6] and obtain a quadrature formula for the improper integral

$$(\Lambda_0 g)(t) := -\frac{1}{2\pi}\int_0^{2\pi} \log\left(\frac{4}{e}\sin^2\frac{t-s}{2}\right) g(s)\,\mathrm{d}s \tag{13}$$

by replacing the function $g(s)$ by its trigonometric interpolation polynomial

$$(\mathcal{P}_n g)(s) := \sum_{j=0}^{2n-1} g\left(\frac{j\pi}{n}\right) L_j(s),$$

where the Lagrange basis is given by

$$L_j(s) := \frac{1}{2n}\left(1 + 2\sum_{k=1}^{n-1}\cos k\left(s-\frac{j\pi}{n}\right) + \cos n\left(s-\frac{j\pi}{n}\right)\right) \quad \forall j = 0,\ldots,2n-1.$$

We then obtain

$$(\Lambda_0 g)(t) \simeq \tilde{\mathcal{Q}}_n g(t) := \sum_{j=0}^{2n-1} R_j^{(n)}(t)\, g\left(\frac{j\pi}{n}\right),$$

where, for $j = 0,\ldots,2n-1$, the weights

$$R_j^{(n)}(t) = \frac{1}{2n} + \frac{1}{n}\sum_{m=1}^{n-1}\frac{1}{m}\cos m\left(t-\frac{j\pi}{n}\right) + \frac{1}{2n^2}\cos n\left(t-\frac{j\pi}{n}\right)$$

are deduced by evaluating explicitly the integrals $(\Lambda_0 L_j)(t)$; cf. [6].

Using the splitting

$$V(t,s) = -\frac{1}{2\pi}V_1(t,s)\log\left(\frac{4}{e}\sin^2\frac{t-s}{2}\right) + V_2(t,s), \tag{14}$$

of the single layer acoustic potential kernel, where $V_1(t,s) := \frac{1}{2}J_0(k|\mathbf{x}(t)-\mathbf{x}(s)|)$ and J_0 is the Bessel function of order zero, we obtain

$$\rho_f\omega^2 c(\xi,\eta) = \int_0^{2\pi}\Lambda_0(V_1(t,\cdot)\xi(\cdot))(t)\eta(t)\,\mathrm{d}t + \int_0^{2\pi}\left(\int_0^{2\pi}V_2(t,s)\xi(s)\,\mathrm{d}s\right)\eta(t)\,\mathrm{d}t. \tag{15}$$

Hereafter, taking into account that V_1 and V_2 are in $\mathscr{C}^\infty_{2\pi}$ with respect to each variable, the first term of the right-hand side in (15) may be approximated by using the quadrature rule $\tilde{\mathcal{Q}}_n$ for the internal integral and $\mathcal{Q}_n$ for the external one. The two-dimensional quadrature rule derived from $\mathcal{Q}_n$ is applied to the second term. In other words, we are introducing an approximation of the sesquilinear form $c(\cdot,\cdot)$ on $T_n\times T_n$ given by

$$\rho_f\omega^2 c_n(\xi,\eta) := \mathcal{Q}_n[\tilde{\mathcal{Q}}_n[V_1(t,\cdot)\xi(\cdot)]\eta(t)] + \mathcal{Q}_n[\mathcal{Q}_n[V_2(t,\cdot)\xi(\cdot)]\eta(t)].$$

Notice that we may equivalently write

$$c_n(\xi,\eta) = \frac{1}{\rho_f\omega^2}\sum_{i=0}^{2n-1}\left(\sum_{j=0}^{2n-1} C_{ij}\xi\left(\frac{j\pi}{n}\right)\right)\eta\left(\frac{i\pi}{n}\right)$$

with

$$C_{ij} := \frac{\pi}{n}R_j^{(n)}\left(\frac{i\pi}{n}\right)V_1\left(\frac{i\pi}{n},\frac{j\pi}{n}\right) + \frac{\pi^2}{n^2}V_2\left(\frac{i\pi}{n},\frac{j\pi}{n}\right).$$

5.3. Approximation of $b(\cdot,\cdot)$

We point out that the kernel $K(\cdot,\cdot)$ associated to the sesquilinear form $b(\cdot,\cdot)$ is continuous but not differentiable, therefore, it is necessary to split it, as we did for $V(\cdot,\cdot)$ in (14), before using any quadrature rule. Here again, we follow [6] and write

$$K(t,s) = -\frac{1}{2\pi}K_1(t,s)\log\left(\frac{4}{e}\sin^2\frac{t-s}{2}\right) + K_2(t,s) \tag{16}$$

with

$$K_1(t,s) := -\frac{k}{2}J_1(k|\mathbf{x}(t)-\mathbf{x}(s)|)\frac{x_2'(s)(x_1(t)-x_1(s)) - x_1'(s)(x_2(t)-x_2(s))}{|\mathbf{x}(t)-\mathbf{x}(s)|},$$

and J_1 being the Bessel function of order one. It turns out that K_1 and K_2 belong to $\mathscr{C}^\infty_{2\pi}$ in each variable.

We introduce the composite trapezoidal rule

$$\mathscr{Q}_N(g) := \frac{\pi}{N}\sum_{i=0}^{2N-1} g\left(\frac{i\pi}{N}\right)$$

associated to the uniform partition of $[0,2\pi]$ into $2N$ grid points. Given $q \in V_h^f$ and $\eta \in T_n$, our strategy consists in approximating

$$\begin{aligned}\omega^2\rho_f b(\gamma_h q,\eta) = \frac{1}{2}\int_0^{2\pi}\gamma_h q(t)\eta(t)\,\mathrm{d}t - \int_0^{2\pi}\Lambda_0(K_1(\cdot,s)\eta(\cdot))(s)\gamma_h q(s)\,\mathrm{d}s \\ - \int_0^{2\pi}\left(\int_0^{2\pi}K_2(t,s)\gamma_h q(s)\,\mathrm{d}s\right)\eta(t)\,\mathrm{d}t\end{aligned}$$

by employing $\mathscr{Q}_n$, $\tilde{\mathscr{Q}}_n$ and $\mathscr{Q}_N$ as follows:

$$\omega^2\rho_f b_{h,n}(\gamma_h q,\eta) := \frac{1}{2}\int_0^{2\pi}\gamma_h q(t)\eta(t)\,\mathrm{d}t - \mathscr{Q}_N[\tilde{\mathscr{Q}}_n[K_1(\cdot,s)\eta(\cdot)]\gamma_h q(s)] - \mathscr{Q}_N[\mathscr{Q}_n[K_2(\cdot,s)\eta(\cdot)]\gamma_h q(s)].$$

In other words,

$$b_{h,n}(\gamma_h q,\eta) = \frac{1}{\omega^2\rho_f}\sum_{j=0}^{2N-1}\left(\sum_{i=0}^{2n-1}B_{ij}\eta\left(\frac{i\pi}{n}\right)\right)q\left(\frac{j\pi}{N}\right),$$

where

$$B_{ij} := \frac{1}{2}\int_0^{2\pi} \ell_j(t)L_i(t)\,\mathrm{d}t - \frac{\pi}{N}R_i^{(n)}\left(\frac{j\pi}{N}\right)K_1\left(\frac{i\pi}{n},\frac{j\pi}{N}\right) + \frac{\pi^2}{nN}K_2\left(\frac{i\pi}{n},\frac{j\pi}{N}\right)$$

and $\{\ell_j(t),\ j=0,\dots,2N-1\}$ is the nodal basis of T_h, i.e., the 2π-periodic, continuous and piecewise linear functions that satisfy $\ell_j(\frac{i\pi}{N}) = \delta_{ij}$ for all $0 \leqslant i,j \leqslant 2N-1$.

5.4. Approximation of $c((\cdot)',(\cdot)')$

Let p and q be two given functions in V_h^f. It is straightforward to show that

$$\omega^2\rho_f c((\gamma_h p)',(\gamma_h q)') = \frac{N^2}{\pi^2}\sum_{i,j=0}^{2N-1}(\beta_{i,j} - \beta_{i+1,j} - \beta_{i,j+1} + \beta_{i+1,j+1})p\left(\frac{j\pi}{N}\right)q\left(\frac{i\pi}{N}\right),$$

where

$$\beta_{i,j} := \int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} V(t,s)\,\mathrm{d}s\,\mathrm{d}t$$

with $t_i = \frac{i\pi}{N}$, $i=0,\dots,2N-1$.

Before defining quadrature rules to approximate the integrals $\beta_{i,j}$ we have to introduce a new decomposition of the singular kernel $V(t,s)$ that is more convenient for our purpose. Namely, we set

$$V(t,s) := V_1(t,s)\log(t-s)^2 + F(t,s)$$

with

$$F(t,s) := V_1(t,s)\log\left(\frac{\frac{4}{e}\sin\frac{t-s}{2}}{t-s}\right)^2 + V_2(t,s).$$

It results that F is $\mathscr{C}^\infty$ in the domain $\mathcal{O} := \{(t,s); |t-s| < 2\pi\}$. Numerical quadratures must then be handled with care in order to avoid the neighborhood of the singular points situated on the lines $\{(t,s); |t-s| = 2\pi\}$. In fact, it suffices to compute the approximations $\tilde{\beta}_{i,j}$ of $\beta_{i,j}$ for indices that satisfy $|i-j| \leqslant N$ and recover $\tilde{\beta}_{i,j}$ for $i,j = 0,\dots,2N-1$ by taking advantage of the 2π periodicity of $V(\cdot,\cdot)$ in both variables. Hence, for $0 \leqslant i \leqslant 2N-1$ and $i-N \leqslant j \leqslant i+N-1$ we define

$$\tilde{\beta}_{i,j} := V_1(t_{i+1/2},t_{j+1/2})\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} \log(t-s)^2\,\mathrm{d}s\,\mathrm{d}t + \frac{\pi^2}{N^2}F(t_{i+1/2},t_{j+1/2}),$$

where $t_{i+1/2} := (i+\frac{1}{2})\frac{\pi}{N}$. The integral appearing in the definition of $\tilde{\beta}_{i,j}$ is computed exactly. We also point out that we used the two-dimensional midpoint formula

$$\int_{t_i}^{t_{i+1}}\int_{t_j}^{t_{j+1}} F(t,s)\,\mathrm{d}s\,\mathrm{d}t \simeq \frac{\pi^2}{N^2}F(t_{i+1/2},t_{j+1/2}).$$

It follows that our approximation of $c((\cdot)',(\cdot)')$ on $T_h \times T_h$ is given by

$$c_h((\gamma_h p)',(\gamma_h q)') := \frac{1}{\omega^2\rho_f}\sum_{i=0}^{2N-1}\sum_{j=-N+i}^{N+i-1} E_{ij}p\left(\frac{j\pi}{N}\right)q\left(\frac{i\pi}{N}\right),$$

where

$$\mathrm{E}_{ij} := \frac{N^2}{\pi^2}(\tilde{\beta}_{i,j} - \tilde{\beta}_{i+1,j} - \tilde{\beta}_{i,j+1} + \tilde{\beta}_{i+1,j+1}).$$

5.5. Approximation of $d(\cdot,\cdot)$

For p and q in V_h^f, we have the decomposition

$$\omega^2\rho_f d(\gamma_h p, \gamma_h q) = \int_0^{2\pi}\int_0^{2\pi}\tilde{V}_1(t,s)\log(t-s)^2\gamma_h p\gamma_h q\,\mathrm{d}s\,\mathrm{d}t + \int_0^{2\pi}\int_0^{2\pi}\tilde{F}(t,s)\gamma_h p\gamma_h q\,\mathrm{d}s\,\mathrm{d}t,$$

where $\tilde{V}_1(t,s) = \mathbf{x}'(t)\cdot\mathbf{x}'(s)V_1(t,s)$ and $\tilde{F}(t,s) = \mathbf{x}'(t)\cdot\mathbf{x}'(s)F(t,s)$.

We propose an approximation $d_h(u,v)$ of $d(u,v)$ defined by

$$d_h(\gamma_h p, \gamma_h q) := \frac{1}{\omega^2\rho_f}\sum_{i=0}^{2N-1}\sum_{j=-N+i}^{N+i-1} D_{ij}\,p\left(\frac{j\pi}{N}\right)q\left(\frac{i\pi}{N}\right)$$

with

$$D_{ij} := \tilde{V}_1\left(\frac{i\pi}{N},\frac{j\pi}{N}\right)\int_{t_{i-1}}^{t_{i+1}}\int_{t_{i-1}}^{t_{i+1}}\log(t-s)^2\ell_j(s)\ell_i(t)\,\mathrm{d}s\,\mathrm{d}t + \frac{\pi^2}{N^2}\tilde{F}\left(\frac{i\pi}{N},\frac{j\pi}{N}\right).$$

The first term of the right-hand side of the last equation is computed exactly. We also notice that we approximated the integral of $\tilde{F}(t,s)\gamma_h p(s)\gamma_h q(t)$ by using the bidimensional trapezoidal rule.

We are now in a position to propose a completely discrete version of the Galerkin scheme (12):

$$\begin{aligned}
&\text{find } \mathbf{u}_h^* \in (V_h^s)^2,\ p_h^* \in V_h^f \text{ and } \xi_n^* \in T_n \text{ such that}\\
&\quad A(\mathbf{u}_h^*, \mathbf{v}) + D(\mathbf{v}, p_h^*) = L_h(\mathbf{v}),\\
&\quad a(p_h^*, q) + D(\mathbf{u}_h^*, q) - b_{h,n}(\gamma_h q, \xi_n^*) + c_h((\gamma_h p_h^*)', (\gamma_h q)') - k^2 d_h(\gamma_h p_h^*, \gamma_h q) = \ell_h(q),\\
&\quad -b_{h,n}(\gamma_h p_h^*, \eta) - c_n(\xi_n^*, \eta) = 0
\end{aligned} \tag{17}$$

for all $\mathbf{v} \in (V_h^s)^2$, $q \in V_h^f$ and $\eta \in T_n$.

5.6. Matrix form of the fully discrete problem

Let us denote by $\{\varphi_i^s,\ i=1,\ldots,M_h^s\}$ and $\{\varphi_i^f,\ i=1,\ldots,M_h^f\}$ the nodal basis of V_h^s and V_h^f, respectively. We also consider the canonical basis $\{\mathbf{e}_1 := (1,0), \mathbf{e}_2 := (0,1)\}$ of $\mathbb{R}^2$.

For $1 \leqslant \alpha, \beta \leqslant 2$ we denote by $\mathbf{A}^{\alpha\beta}$ the $M_h^s \times M_h^s$ matrix whose entries are given by

$$\mathbf{A}_{ij}^{\alpha\beta} := A(\varphi_i^s\mathbf{e}_\alpha, \varphi_j^s\mathbf{e}_\beta).$$

Let us also introduce the $M_h^s \times M_h^f$ matrix $\mathbf{D}^\alpha$ ($\alpha = 1,2$)

$$\mathbf{D}_{ij}^\alpha := D(\varphi_i^s\mathbf{e}_\alpha, \varphi_j^f).$$

If we set

$$\mathbf{u}_h^* = \sum_{\alpha=1}^{2}\sum_{i=1}^{M_h^s}\mathbf{u}_i^{(\alpha)}\varphi_i^s\mathbf{e}_\alpha, \quad p_h^* = \sum_{i=1}^{M_h^f}p_i\varphi_i^f, \quad \xi_n^* = \sum_{i=0}^{2n-1}\xi_i L_i$$

and use the superscript $(\cdot)^{\mathrm{T}}$ to denote transposition of matrices, then, the matricial interpretation of (17) takes the form

$$\begin{pmatrix} \mathbf{A} & \mathbf{D} & \mathbf{0} \\ \mathbf{D}^{\mathrm{T}} & \mathbf{R} & \mathbf{K} \\ \mathbf{0} & \mathbf{K}^{\mathrm{T}} & -\mathbf{C} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{p} \\ \xi \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \\ \mathbf{0} \end{pmatrix}, \tag{18}$$

where

$$\mathbf{A} := \begin{pmatrix} \mathbf{A}^{11} & \mathbf{A}^{12} \\ \mathbf{A}^{21} & \mathbf{A}^{22} \end{pmatrix}, \quad \mathbf{D} := \begin{pmatrix} \mathbf{D}^{1} \\ \mathbf{D}^{2} \end{pmatrix}$$

and

$$\mathbf{R}_{ij} := a(\varphi_i^f, \varphi_j^f) + c_h((\gamma_h\varphi_i^f)', (\gamma_h\varphi_j^f)') - k^2 d_h(\gamma_h\varphi_i^f, \gamma_h\varphi_j^f),$$

$$\mathbf{K}_{ik} := -b_{h,n}(\gamma_h\varphi_i^f, L_k) \quad \text{and} \quad \mathbf{C}_{k\ell} := c_n(L_j, L_\ell).$$

The right-hand side of (18) is given by

$$\mathbf{F} := \begin{pmatrix} \mathbf{F}^{1} \\ \mathbf{F}^{2} \end{pmatrix} \quad \text{with } \mathbf{F}_i^{\alpha} := L_h(\varphi_i^s \mathbf{e}_\alpha) \quad (\alpha = 1,2) \ \ (i = 1, \ldots, M_h^s)$$

and

$$\mathbf{G}_i := \ell_h(\varphi_i^f) \quad (i = 1, \ldots, M_h^f).$$

The matrix in (18) is complex symmetric but it is badly structured since **A**, **D** and the part of **R** corresponding to the sesquilinear form $a(\cdot,\cdot)$ are sparse matrices while **C** and **K** are full. The global matrix is too large to be stored and handled. In the next section we will propose an efficient iterative method to solve (18).

6. Numerical results

We test our numerical method on a problem (1) whose exact solution is known explicitly. We take $\Omega_s = (-0.2, 0.2) \times (-0.4, 0.4)$ and define Γ to be the ellipse centered at the origin with minor and major semiaxes equal to 0.4 and 0.6, respectively. We also choose $\rho_s = \rho_f = c = \lambda = \mu = 1$. Let us denote by K_0, K_1 and K_2 the modified Bessel functions of the second kind and order 0, 1 and 2, respectively. The function given by

$$\mathbf{u}_e(\boldsymbol{x}) = \frac{1}{2\pi} \begin{pmatrix} \psi(\boldsymbol{x}) - \frac{(x_1-0.3)^2}{r_1^2}\chi(\boldsymbol{x}) \\ -\frac{(x_1-0.3)x_2}{r_1^2}\chi(\boldsymbol{x}) \end{pmatrix} \quad \left(r_1 := \sqrt{(x_1-0.3)^2 + x_2^2} \right)$$

with

$$\psi(\boldsymbol{x}) := K_0(\imath\omega r_1) + \frac{1}{\imath\omega r_1}\left(K_1(\imath\omega r_1) - \frac{1}{\sqrt{3}} K_1\left(\frac{\imath\omega r_1}{\sqrt{3}} \right) \right)$$

and

$$\chi(\boldsymbol{x}) := K_2(\imath\omega r_1) - \frac{1}{3} K_2\left(\frac{\imath\omega r_1}{\sqrt{3}} \right)$$

is a solution of the elastodynamic equation in Ω_s when $\mathbf{f} = \mathbf{0}$.

On the other hand, the scalar function

$$p_e(\boldsymbol{x}) = H_0^{(1)}(\omega r) \quad (r = |\mathbf{x}|)$$

solves the Helmholtz equation in Ω_f and satisfies the radiation conditions (2). Thus, $(\mathbf{u}_e, p_e)$ is solution of (1) if we impose on Σ the transmission conditions:

$$\sigma(\mathbf{u})\mathbf{n} + p\mathbf{n} = \sigma(\mathbf{u}_e)\mathbf{n} + p_e\mathbf{n},$$
$$\omega^2\mathbf{u}\cdot\mathbf{n} - \frac{\partial p}{\partial \mathbf{n}} = \omega^2\mathbf{u}_e\cdot\mathbf{n} - \frac{\partial p_e}{\partial \mathbf{n}}.$$

In Table 1, we take $\omega = 1$ and $h = 2\pi/128$ while $\omega = 5$ and $h = 2\pi/128$ in Table 2. In both cases we decrease the spectral parameter n until we obtain the smallest value that preserves the order of accuracy.

We can see that the number of degrees of freedom is drastically reduced. This justifies the following strategy used to solve the linear systems of equations. We eliminate the boundary variable from (18) to obtain the reduced system

$$\begin{pmatrix} \mathbf{A} & \mathbf{D} \\ \mathbf{D}^{\mathrm{T}} & \mathbf{R}^k + \mathbf{K}\mathbf{C}^{-1}\mathbf{K}^{\mathrm{T}} \end{pmatrix}\begin{pmatrix} \mathbf{u} \\ \mathbf{p} \end{pmatrix} = \begin{pmatrix} \mathbf{F} \\ \mathbf{G} \end{pmatrix}. \tag{19}$$

The system of equations (19) is then solved by a preconditioned GMRES method. We use the block diagonal matrix

Table 1
Convergence history and number of iterations of the method for different values of the parameter n when $\omega = 1$ and $h = 2\pi/128$

$2n$	$\Vert\mathbf{u} - \mathbf{u}_h^*\Vert_{1,\Omega_s}$	$\Vert p - p_h^*\Vert_{1,\Omega_f^-}$
64	3.29×10^{-3}	4.27×10^{-3}
32	3.29×10^{-3}	3.29×10^{-3}
16	3.29×10^{-3}	3.24×10^{-3}
8	3.29×10^{-3}	8.09×10^{-3}

Table 2
Convergence history and number of iterations of the method for different values of the parameter n when $\omega = 5$ and $h = 2\pi/128$

$2n$	$\Vert\mathbf{u} - \mathbf{u}_h^*\Vert_{1,\Omega_s}$	$\Vert p - p_h^*\Vert_{1,\Omega_f^-}$
64	4.23×10^{-3}	9.97×10^{-3}
32	4.23×10^{-3}	9.91×10^{-3}
16	4.23×10^{-3}	9.90×10^{-3}
8	7.88×10^{-3}	4.96×10^{-2}

Table 3
Convergence history and number of iterations of the method for different values of the parameter h when $\omega = 5$ and $n = 8$

h	$\Vert\mathbf{u} - \mathbf{u}_h^*\Vert_{1,\Omega_s}$	$\Vert p - p_h^*\Vert_{1,\Omega_s}$	Iterations
$2\pi/32$	2.62×10^{-2}	7.58×10^{-2}	22
$2\pi/64$	8.76×10^{-3}	2.89×10^{-2}	22
$2\pi/128$	4.23×10^{-3}	9.90×10^{-3}	21
$2\pi/256$	1.9×10^{-3}	5.2×10^{-3}	21

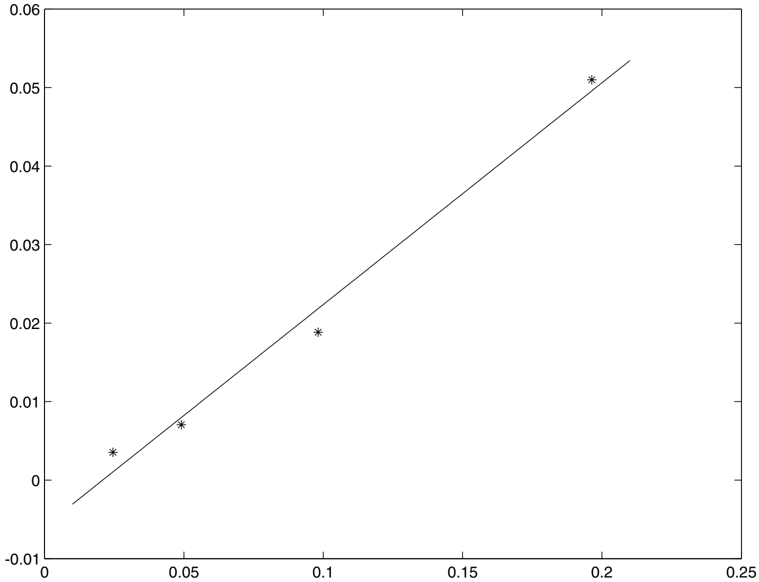

Fig. 2. The arithmetic mean of the H^1-errors in displacement and pressure versus h.

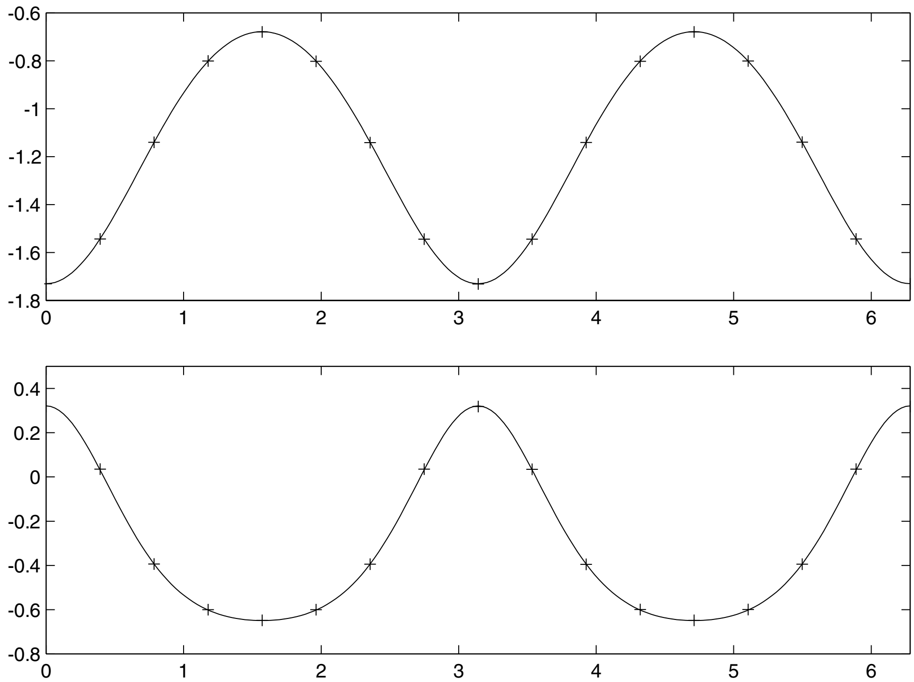

Fig. 3. Real (above) and imaginary (below) parts of the variable ξ. The analytical solution is represented by a line and the computed solution (with $h = 2\pi/128$, $\omega = 5$ and $2n = 16$) by plus signs.

$$\begin{pmatrix} \mathbf{A}_0 & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_0 \end{pmatrix}$$

as a preconditioner, where $\mathbf{A}_0$ and $\mathbf{R}_0$ are the matrices associated to the sesquilinear forms $\int_{\Omega_s} \sigma(\mathbf{u}) : \varepsilon(\mathbf{v})\, \mathrm{d}\boldsymbol{x}$ and $\int_{\Omega_f^-} \nabla p \cdot \nabla q\, \mathrm{d}\mathbf{x}$, respectively. We use a version of GMRES without restarts. We take as an initial guess an identically vanishing function in both Ω_s and Ω_f^-. Iterations are continued until $\|r_{k+1}\|_2/\|r_k\|_2 < 10^{-6}$, where r_k is the kth residual.

Each iteration of the GMRES method entails the solution of a linear system with a full but small matrix $\mathbf{C}$ and the solution of two other linear systems with sparse matrices $\mathbf{A}_0$ and $\mathbf{R}_0$. This can be performed by any of the numerous strategies existing in the literature for these standard stiffness matrices. Table 3 shows the number of iterations against h with $n = 8$ and $\omega = 5$. The numerical results suggest that the method has a number of iterations bounded independently of the critical parameter h. Fig. 2 depicts the results of Table 3 and shows that, as expected, the error grows linearly with respect to the mesh parameter. Finally, the accuracy of our method on the coupling boundary is illustrated by Fig. 3 with the data $h = 2\pi/128$, $\omega = 5$ and $2n = 16$. In each graphic we compare the real and imaginary parts of the 2π-periodic unknown on the boundary to its discrete counterpart. The exact and approximated solutions are superposed in each graphic. The analytical solution is represented by the continuous line.

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